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Thermal stresses in a long cylinder under Gaussian-distributed heating in the framework of fractional thermoelasticity

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An axisymmetric problem for Gaussian-distributed heating of a lateral surface of an infinite cylinder is solved in the framework of fractional thermoelasticity based on the time-fractional heat conduction equation with the Caputo derivative. The representation of stresses in terms of displacement potential and Love function is used to satisfy the boundary conditions on a surface of a cylinder. The results of numerical calculation are presented for different values of the order of fractional derivative and nondimensional time.

Keywords: *fractional thermoelasticity, Caputo derivative, Mittag-Leffler function, thermal stresses*

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1. Introduction

The standard thermoelasticity combines the elasticity theory and the classical heat conduction based on the Fourier law for the heat flux

$$\mathbf{q} = -k \operatorname{grad} T \tag{1}$$

and the parabolic heat conduction equation for temperature

$$\frac{\partial T}{\partial t} = a\,\Delta T. \tag{2}$$

However, many theoretical and experimental studies of transport phenomena testify that in media with complex internal structure the classical Fourier law and parabolic heat conduction equation are no longer accurate enough. Several nonclassical theories have been proposed, in which the Fourier law and the parabolic heat conduction equation are replaced by more complicated equations. In the theory of heat conduction proposed by Gurtin and Pipkin [1], the Fourier law was substituted by the time-nonlocal dependence between the heat flux vector and the temperature gradient. The different choice of the memory kernel results in the corresponding generalized theories of heat conduction. A survey of such generalizations of the Fourier law can be found in [2–5].

Each generalization of the theory of heat conduction results in the corresponding generalization of thermoelasticity. For example, Cattaneo theory [6] of heat conduction with the "short-tail" exponential kernel leads to the telegraph equation for temperature

$$\frac{\partial T}{\partial t} + \zeta\frac{\partial^2 T}{\partial t^2} = a\,\Delta T \tag{3}$$

and the generalized thermoelasticity of Lord and Shulman [7]. Heat conduction with "total" memory is described by the wave equation for temperature [8, 9]

$$\frac{\partial^2 T}{\partial t^2} = a\,\Delta T \tag{4}$$

and creates the basis of thermoelasticity without energy dissipation developed by Green and Naghdi [9].

It was shown in [2, 3, 10, 11] that the "long-tail" memory with the power memory kernel

$$\mathbf{q} = -kD_{RL}^{1-\alpha}\,\mathrm{grad}\,T(t), \quad 0 < \alpha \leqslant 1, \tag{5}$$

$$\mathbf{q} = -kI^{\alpha-1}\,\mathrm{grad}\,T(t), \quad 1 < \alpha \leqslant 2, \tag{6}$$

in combination with the energy equation results in the time-fractional heat conduction equation

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\,\Delta T. \tag{7}$$

Here $I^\alpha f(t)$ and $D_{RL}^\alpha f(t)$ are the Riemann-Liouville fractional integral and derivative of order α, respectively, $\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha}$ is the Caputo derivative [12–14]:

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau, \qquad \alpha > 0, \tag{8}$$

$$D_{RL}^\alpha f(t) = \frac{\mathrm{d}^m}{\mathrm{d}t^m}\left[\frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f(\tau)\,\mathrm{d}\tau\right], \quad m-1 < \alpha < m, \tag{9}$$

$$\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha} = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{\mathrm{d}^m f(\tau)}{\mathrm{d}\tau^m}\,\mathrm{d}\tau, \quad m-1 < \alpha < m, \tag{10}$$

where $\Gamma(\alpha)$ is the gamma function.

The thermoelasticity theory based on equation (7) was proposed in [10]. This approach was developed in succeeding papers [2–5, 15–24] and was reviewed in the book [25].

The time-fractional heat conduction equation in cylindrical coordinates was studied in [26–37]. In this paper we will investigate axisymmetric thermal stresses in an infinite cylinder with the Gaussian-distributed boundary value of temperature on its lateral surface.

2. Statement of the problem

A quasi-static uncoupled theory of thermal stresses based on time-fractional heat conduction equation is governed by the following system of equations [2, 10, 24, 25]:

- the equilibrium equation in terms of displacements

$$\mu\Delta\mathbf{u} + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} = \beta_T K_T\,\mathrm{grad}\,T, \tag{11}$$

- the stress-strain-temperature relation

$$\boldsymbol{\sigma} = 2\mu\mathbf{e} + (\lambda\,\mathrm{tr}\,\mathbf{e} - \beta_T K_T T)\,\mathbf{I}, \tag{12}$$

- the time-fractional heat conduction equation

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right), \qquad 0 \leqslant r < R, \quad -\infty < z < \infty. \tag{13}$$

Here $\boldsymbol{\sigma}$ is the stress tensor, $\mathbf{e}$ is the strain tensor, λ and μ are Lamé constants, $K_T = \lambda + 2\mu/3$ is the modulus of dilatation, β_T is the thermal coefficient of volumetric expansion, a is the thermal diffusivity coefficient, $\mathbf{I}$ denotes the unit tensor.

We assume the zero initial conditions

$$t=0: \qquad T=0, \qquad 0<\alpha\leqslant 2, \tag{14}$$

$$t=0: \qquad \frac{\partial T}{\partial t}=0, \qquad 1<\alpha\leqslant 2, \tag{15}$$

and the Dirichlet boundary condition corresponding to the Gaussian-distributed heating of a surface

$$r=R: \quad T=\frac{w_0}{\gamma\sqrt{2\pi}}\exp\left(-\frac{z^2}{2\gamma^2}\right). \tag{16}$$

The surface of a cylinder is traction free:

$$r=R: \quad \sigma_{rr}=0, \qquad \sigma_{rz}=0. \tag{17}$$

In a quasi-static statement of the thermoelasticity problem initial values of mechanical quantities are not considered. The zero conditions at infinity

$$\lim_{z\to\pm\infty}\mathbf{u}(r,z,t)=0, \qquad \lim_{z\to\pm\infty}T(r,z,t)=0 \tag{18}$$

are also assumed.

3. Solution of the heat conduction equation

Application of the Laplace transform with respect to time t and the exponential Fourier transfom with respect to the axial coordinate z to the fractional heat conduction equation (13) and the boundary condition (16) give

$$s^\alpha\widetilde{T}^*=a\left(\frac{\partial^2\widetilde{T}^*}{\partial r^2}+\frac{1}{r}\frac{\partial\widetilde{T}^*}{\partial r}-\eta^2\widetilde{T}^*\right), \tag{19}$$

$$r=R: \quad \widetilde{T}^*=\frac{w_0}{s\sqrt{2\pi}}\exp\left(-\frac{\gamma^2\eta^2}{2}\right), \tag{20}$$

where the asterisk denotes the Laplace transform, the tilde marks the Fourier transform, s is the Laplace transform variable, η is the Fourier transform variable. In (19) we have used the following Laplace transform rule for the Caputo derivative [12–14]

$$\mathcal{L}\left\{\frac{\mathrm{d}f(t)}{\mathrm{d}t^\alpha}\right\}=s^\alpha f^*(s)-\sum_{k=0}^{m-1}f^{(k)}(0^+)s^{\alpha-1-k}, \quad m-1<\alpha<m. \tag{21}$$

Next, we use the finite Hankel transform with respect to the radial coordinate r [38]. Recall that for Dirichlet boundary condition this transform has the following form:

$$\mathcal{H}\{f(r)\}=\widehat{f}(\xi_k)=\int_0^R f(r)\,J_0(r\xi_k)\,r\,\mathrm{d}r, \tag{22}$$

$$\mathcal{H}^{-1}\{\widehat{f}(\xi_k)\}=f(r)=\frac{2}{R^2}\sum_{k=1}^{\infty}\widehat{f}(\xi_k)\frac{J_0(r\xi_k)}{[J_1(R\xi_k]^2} \tag{23}$$

with the sum over all positive roots of the zeroth-order Bessel function

$$J_0(R\xi_k)=0. \tag{24}$$

In this case

$$\mathcal{H}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right\}=-\xi_k^2\widehat{f}(\xi_k)+R\xi_k J_1(R\xi_k)f(R). \tag{25}$$

Taking into account (20) and (25), from (19) we obtain the solution in the transform domain:

$$\widehat{\widetilde{T}}^*(\xi_k,\eta,s)=\frac{aR\,w_0\,\xi_k}{\sqrt{2\pi}}\,J_1(R\xi_k)\exp\left(-\frac{\gamma^2\eta^2}{2}\right)\frac{1}{s\left[s^\alpha+a(\xi_k^2+\eta^2)\right]} \tag{26}$$

or

$$\widehat{\widetilde{T}}^*(\xi_k,\eta,s)=\frac{R\,w_0}{\sqrt{2\pi}}\,\frac{\xi_k\,J_1(R\xi_k)}{\xi_k^2+\eta^2}\,\exp\left(-\frac{\gamma^2\eta^2}{2}\right)\left[\frac{1}{s}-\frac{s^\alpha}{s^\alpha+a(\xi_k^2+\eta^2)}\right]. \tag{27}$$

To inverse the Laplace transform the following formula [12–14]

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^\alpha+b}\right\}=E_\alpha(-bt^\alpha) \tag{28}$$

will be used. Here $E_\alpha(z)$ is the Mittag-Leffler function in one parameter α [12–14, 40]:

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)},\qquad \alpha>0,\quad z\in C, \tag{29}$$

being a generalization of the exponential function

$$\mathrm{e}^z=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n+1)},\qquad z\in C. \tag{30}$$

Inversion of the Laplace transform gives:

$$\widehat{\widetilde{T}}(\xi_k,\eta,t)=\frac{R\,w_0}{\sqrt{2\pi}}\,\frac{\xi_k\,J_1(R\xi_k)}{\xi_k^2+\eta^2}\,\exp\left(-\frac{\gamma^2\eta^2}{2}\right)\left\{1-E_\alpha\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right]\right\}. \tag{31}$$

Inversion of the Fourier and finite Hankel transforms allows us to arrive at the expression for the temperature field in a cylinder

$$T(r,z,t)=\frac{w_0}{R\pi}\int_{-\infty}^{\infty}\sum_{k=1}^{\infty}\frac{\xi_k J_0(r\xi_k)}{\left(\xi_k^2+\eta^2\right)J_1(R\xi_k)}\left\{1-E_\alpha\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right]\right\}$$
$$\times\exp\left(-\frac{\gamma^2\eta^2}{2}\right)\cos(z\eta)\,\mathrm{d}\eta. \tag{32}$$

It is worth noting that the relation [38]

$$\frac{2}{R}\sum_{k=1}^{\infty}\frac{\xi_k J_0(r\xi_k)}{\left(\xi_k^2-\beta^2\right)J_1(R\xi_k)}=\frac{J_0(r\beta)}{J_0(R\beta)} \tag{33}$$

for $\beta=i\eta$ can be rewritten as

$$\frac{2}{R}\sum_{k=1}^{\infty}\frac{\xi_k J_0(r\xi_k)}{\left(\xi_k^2+\eta^2\right)J_1(R\xi_k)}=\frac{I_0(r\eta)}{I_0(R\eta)}, \tag{34}$$

where $I_n(r)$ is the modified Bessel function of order n.

Hence (32) takes the form

$$T(r,z,t) = \frac{w_0}{2\pi} \int_{-\infty}^{\infty} \frac{I_0(r\eta)}{I_0(R\eta)} \exp\left(-\frac{\gamma^2\eta^2}{2}\right) \cos(z\eta)\,\mathrm{d}\eta$$
$$- \frac{w_0}{R\pi} \int_{-\infty}^{\infty} \sum_{k=1}^{\infty} E_\alpha\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right] \frac{\xi_k J_0(r\xi_k)}{\left(\xi_k^2+\eta^2\right) J_1(R\xi_k)} \exp\left(-\frac{\gamma^2\eta^2}{2}\right) \cos(z\eta)\,\mathrm{d}\eta. \quad (35)$$

At the boundary surface $r = R$, the first integral in (35) satisfies the boundary condition (16), whereas the second one equals zero due to (24).

4. Investigation of thermal stresses

As in the classical thermoelasticity [40, 41] we can use the representation of the stress tensor in terms of the displacement potential Φ. The part of stresses due to the displacement potential describes the influence of the temperature field and satisfies the equation

$$\Delta\Phi = mT, \qquad m = \frac{1+\nu}{1-\nu}\frac{\beta_T}{3}. \quad (36)$$

In cylindrical coordinates in the axisymmetric case

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{\partial^2\Phi}{\partial z^2} = mT \quad (37)$$

and

$$\sigma_{rr}^{(1)} = 2\mu\left(\frac{\partial^2\Phi}{\partial r^2} - \Delta\Phi\right), \quad (38)$$
$$\sigma_{\theta\theta}^{(1)} = 2\mu\left(\frac{1}{r}\frac{\partial\Phi}{\partial r} - \Delta\Phi\right), \quad (39)$$
$$\sigma_{zz}^{(1)} = 2\mu\left(\frac{\partial^2\Phi}{\partial z^2} - \Delta\Phi\right), \quad (40)$$
$$\sigma_{rz}^{(1)} = 2\mu\frac{\partial^2\Phi}{\partial r\partial z}. \quad (41)$$

After applying the Fourier and finite Hankel transforms to equation (36) it assumes the form

$$\widehat{\widetilde{\Phi}} = -\frac{m}{\xi_k^2+\eta^2}\widehat{\widetilde{T}} \quad (42)$$

or taking into account (31)

$$\widehat{\widetilde{\Phi}} = -\frac{mR\,w_0}{\sqrt{2\pi}}\frac{\xi_k\,J_1(R\xi_k)}{(\xi_k^2+\eta^2)^2} \exp\left(-\frac{\gamma^2\eta^2}{2}\right)\left\{1 - E_\alpha\left[-a\left(\xi_k^2+\eta^2\right)t^\alpha\right]\right\}, \quad (43)$$

which gives

$$\widetilde{\Phi} = -C\exp\left(-\frac{\gamma^2\eta^2}{2}\right)\sum_{k=1}^{\infty} P_k \frac{J_0(r\xi_k)}{J_1(R\xi_k)}, \quad (44)$$

where

$$C = \frac{\sqrt{2} m w_0}{R\sqrt{\pi}}, \tag{45}$$

$$P_k = \frac{\xi_k}{\left(\xi_k^2 + \eta^2\right)^2} \left\{1 - E_\alpha \left[-a\left(\xi_k^2 + \eta^2\right) t^\alpha\right]\right\}. \tag{46}$$

The components of the stress tensor $\boldsymbol{\sigma}^{(1)}$ (38)–(41) in the Fourier transform domain are expressed as

$$\widetilde{\sigma}_{rr}^{(1)} = -2\mu C \exp\left(-\frac{\gamma^2\eta^2}{2}\right) \sum_{k=1}^{\infty} P_k \frac{\xi_k J_1(r\xi_k) + r\eta^2 J_0(r\xi_k)}{r J_1(R\xi_k)}, \tag{47}$$

$$\widetilde{\sigma}_{\theta\theta}^{(1)} = 2\mu C \exp\left(-\frac{\gamma^2\eta^2}{2}\right) \sum_{k=1}^{\infty} P_k \frac{\xi_k J_1(r\xi_k) - r\left(\xi_k^2 + \eta^2\right) J_0(r\xi_k)}{r J_1(R\xi_k)}, \tag{48}$$

$$\widetilde{\sigma}_{zz}^{(1)} = -2\mu C \exp\left(-\frac{\gamma^2\eta^2}{2}\right) \sum_{k=1}^{\infty} P_k \frac{\xi_k^2 J_0(r\xi_k)}{J_1(R\xi_k)}, \tag{49}$$

$$\widetilde{\sigma}_{rz}^{(1)} = -2\mu i C\eta \exp\left(-\frac{\gamma^2\eta^2}{2}\right) \sum_{k=1}^{\infty} P_k \frac{\xi_k J_1(r\xi_k)}{J_1(R\xi_k)}. \tag{50}$$

The stress tensor $\boldsymbol{\sigma}^{(1)}$ does not satisfy the traction free condition (17). The part of the stress field expressed in terms of the biharmonic Love function L [41]:

$$\sigma_{rr}^{(2)} = 2\mu \frac{\partial}{\partial z}\left[\nu \Delta L - \frac{\partial^2 L}{\partial r^2}\right], \tag{51}$$

$$\sigma_{\theta\theta}^{(2)} = 2\mu \frac{\partial}{\partial z}\left[\nu \Delta L - \frac{1}{r}\frac{\partial L}{\partial r}\right], \tag{52}$$

$$\sigma_{zz}^{(2)} = 2\mu \frac{\partial}{\partial z}\left[(2-\nu)\Delta L - \frac{\partial^2 L}{\partial z^2}\right], \tag{53}$$

$$\sigma_{rz}^{(2)} = 2\mu \frac{\partial}{\partial r}\left[(1-\nu)\Delta L - \frac{\partial^2 L}{\partial z^2}\right]. \tag{54}$$

allows us to satisfy the prescribed boundary conditions for the components of the total stress tensor $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)}$:

$$r = R: \quad \sigma_{rr}^{(1)} + \sigma_{rr}^{(2)} = 0, \tag{55}$$

$$r = R: \quad \sigma_{rz}^{(1)} + \sigma_{rz}^{(2)} = 0. \tag{56}$$

The biharmonic equation for the Love function

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)^2 L = 0, \qquad 0 \leqslant r < R, \quad -\infty < z < \infty, \tag{57}$$

in the Fourier transform domain is rewritten as

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \eta^2\right)^2 \widetilde{L} = 0 \tag{58}$$

and has the following solution finite at $r = 0$ [38]:

$$\widetilde{L} = A(\eta) I_0(r\eta) + B(\eta) r\eta I_1(r\eta) \tag{59}$$

with $A(\eta)$ and $B(\eta)$ being the integration coefficients.

From (51)–(54) and (59) we get

$$\widetilde{\sigma}_{rr}^{(2)} = 2\mu i\Big\{\eta^3 A(\eta)\left[I_0(r\eta) - (r\eta)^{-1} I_1(r\eta)\right] + \eta^3 B(\eta)\left[(1-2\nu)I_0(r\eta) + r\eta I_1(r\eta)\right]\Big\}, \tag{60}$$

$$\widetilde{\sigma}_{\theta\theta}^{(2)} = 2\mu i\Big[\eta^2 A(\eta) r^{-1} I_1(r\eta) + (1-2\nu)\eta^3 B(\eta) I_0(r\eta)\Big], \tag{61}$$

$$\widetilde{\sigma}_{zz}^{(2)} = -2\mu i\Big\{\eta^3 A(\eta) I_0(r\eta) + \eta^3 B(\eta)\Big[2(2-\nu)I_0(r\eta) + r\eta I_1(r\eta)\Big]\Big\}, \tag{62}$$

$$\widetilde{\sigma}_{rz}^{(2)} = 2\mu\Big\{\eta^3 A(\eta) I_1(r\eta) + \eta^3 B(\eta)\Big[2(1-\nu)I_1(r\eta) + r\eta I_0(r\eta)\Big]\Big\}, \tag{63}$$

which allows us to determine the integration constants

$$A(\eta) = \frac{iC}{\eta^2 D(\eta)} \exp\left(-\frac{\gamma^2\eta^2}{2}\right)\Big\{\left[R^2\eta^2 + 2(1-\nu)\right] I_1(R\eta) + 2(1-\nu)R\eta I_0(R\eta)\Big\}\sum_{k=1}^{\infty}\xi_k P_k, \tag{64}$$

$$B(\eta) = -\frac{iC}{\eta^2 D(\eta)} \exp\left(-\frac{\gamma^2\eta^2}{2}\right) R\eta I_0(R\eta)\sum_{k=1}^{\infty}\xi_k P_k, \tag{65}$$

where

$$D(\eta) = \left[R^2\eta^2 + 2(1-\nu)\right] I_1^2(R\eta) - R^2\eta^2 I_0^2(R\eta). \tag{66}$$

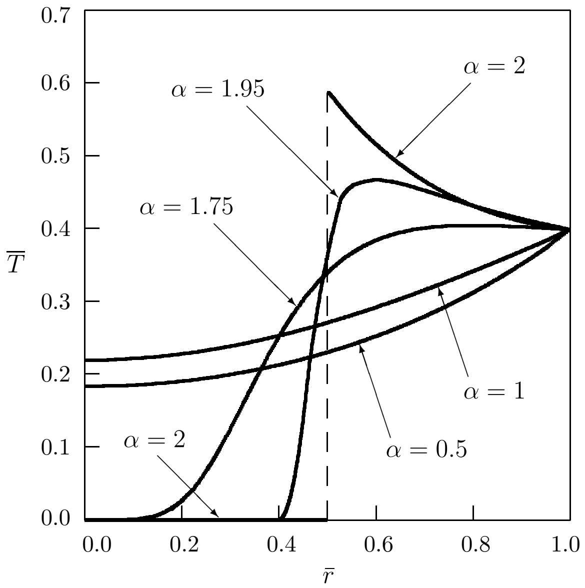

Fig. 1. Dependence of temperature on distance for $\bar{z} = 0$, $\kappa = 0.5$, $\bar{\gamma} = 1$ and different values of α.

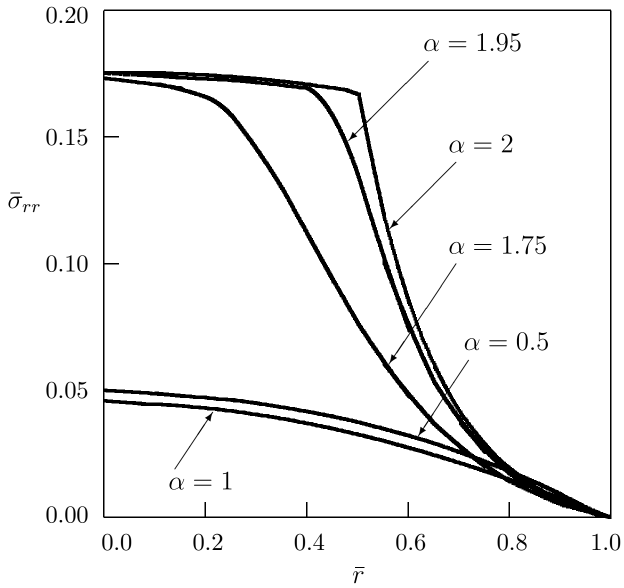

Fig. 2. Dependence of the stress component σ_{rr} on distance for $\bar{z} = 0$, $\kappa = 0.5$, $\bar{\gamma} = 1$ and different values of α.

The results of numerical calculations are shown in Figs. 1–6. The following nondimensional quantities

$$\overline{T} = \frac{RT}{w_0}, \qquad \bar{\sigma}_{ij} = \frac{R\sigma_{ij}}{2\mu m w_0}, \qquad \bar{r} = \frac{r}{R}, \qquad \bar{z} = \frac{z}{R}, \qquad \kappa = \frac{\sqrt{a}t^{\alpha/2}}{R}, \qquad \bar{\gamma} = \frac{\gamma}{R} \tag{67}$$

have been introduced; in calculations we have assumed $\nu = 0.25$. To evaluate the Mittag-Leffler function $E_\alpha(z)$ we have used the algorithm suggested in [42]. Figures 1 and 2 show the dependence of temperature and the stress component σ_{rr} on distance for various values of order of the fractional derivative. Figures 3 and 4 present the dependence of the solution on distance for various values of the

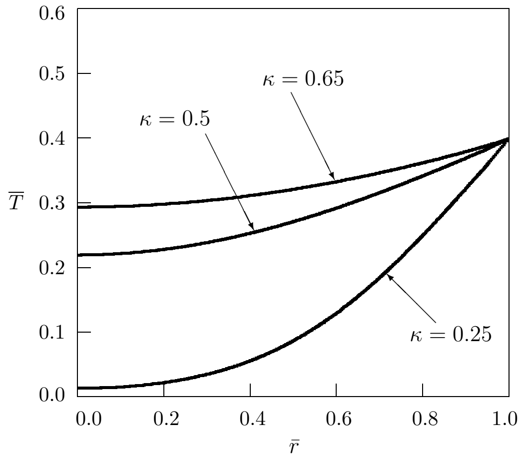

Fig. 3. Dependence of temperature on distance for $\bar{z} = 0$, $\alpha = 1$, $\bar{\gamma} = 1$ and different values of κ.

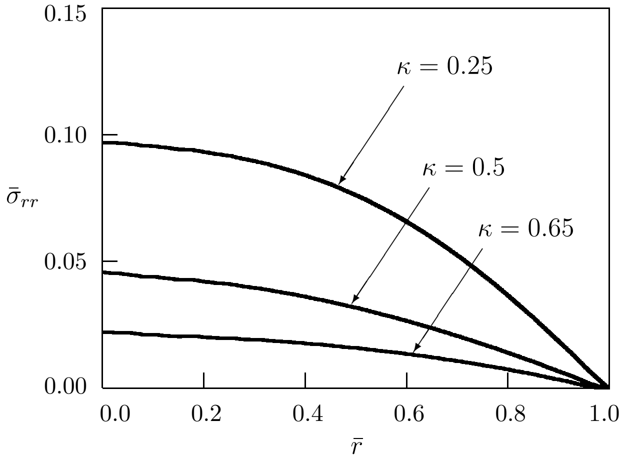

Fig. 4. Dependence of the stress component σ_{rr} on distance for $\bar{z} = 0$, $\alpha = 1$, $\bar{\gamma} = 1$ and different values of κ.

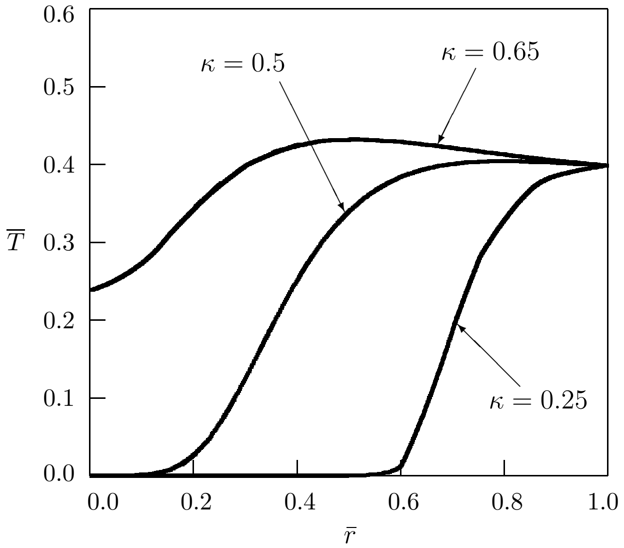

Fig. 5. Dependence of temperature on distance for $\bar{z} = 0$, $\alpha = 1.75$, $\bar{\gamma} = 1$ and different values of κ.

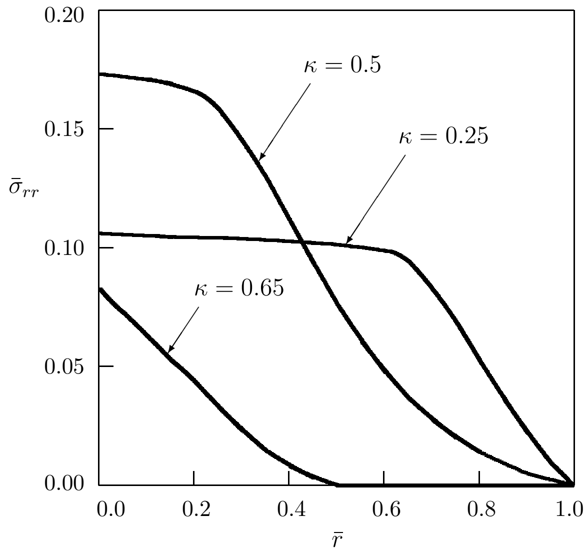

Fig. 6. Dependence of the stress component σ_{rr} on distance for $\bar{z} = 0$, $\alpha = 1.75$, $\bar{\gamma} = 1$ and different values of κ.

parameter κ obtained in the framework of the classical thermoelasticity, whereas Figs. 5 and 6 depict the corresponding results for fractional thermoelasticity with $\alpha = 1.75$.

5. Conclusions

In the case $0 < \alpha < 1$, the time-fractional heat conduction equation with the Caputo derivative of order α interpolates the elliptic Helmholtz equation for temperature ($\alpha = 0$) and the parabolic heat conduction equation ($\alpha = 1$). The associated fractional thermoelasticity interpolates the so-called "localized thermoelasticity" [25] and the classical thermoelasticity. In the case $1 < \alpha < 2$, the time-fractional heat conduction equation interpolates the standard heat conduction equation ($\alpha = 1$) and the hyperbolic wave equation ($\alpha = 2$); the considered theory interpolates the classical theory of thermal stresses and thermoelasticity without energy dissipation introduced by Green and Naghdi [9]. Figure 1 shows how the wave front appearing in the case of the wave equation is approximated by the solution of the fractional heat conduction equation with α approaching 2. In the case of classical thermoelasticity both temperature and the stress component σ_{rr} change with nondimensional time κ monotonically (see Figs. 3 and 4). The nonmonotonic character of such a dependence of the stress component σ_{rr} for $\alpha = 1.75$ is evident from Fig. 6. Fractional thermoelasticity offers considerable possibilities for better describing thermal stresses in porous materials, random and disordered media, fractals, and other solids with complicated internal structure.

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Теплові напруження в довгому циліндрі під час нагрівання його бокової поверхні за гауссовим розподілом у межах дробової термопружності

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Осесиметрична задача про гауссів нагрів бічної поверхні нескінченного циліндра розв'язується в межах дробової термопружності, що базується на рівнянні теплопровідності з дробовою похідною Капуто по часу. Подання напружень через потенціал переміщення і функцію Лява використано для задоволення граничних умов на поверхні циліндра. Результати обчислень подані для різних значень порядку дробової похідної і безвимірного часу.

Ключові слова: *дробова термопружність, похідна Капуто, теплові напруження, функція Міттаг-Леффлера*

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